3.1 (a) Let (V_1, m_1) and (V_2, m_2) be two inner product spaces (i.e. $m_i : V_i \times V_i \to \mathbb{R}$ is symmetric and bilinear, but we do not assume that it is non-degenerate). Prove that there exists a unique inner product $m \doteq m_1 \otimes m_2$ on $V_1 \otimes V_2$ with the property that

$$m(X_1 \otimes X_2, Y_1 \otimes Y_2) = m_1(X_1, Y_1) \cdot m_2(X_2, Y_2).$$

(b) Let (V, m) be an inner product space with a non-degenerate inner product m. Prove that m can be extended to a unique non-degenerate inner product on the space of tensors of type (k, ℓ) over V (i.e. the space $\otimes^k V \otimes^\ell V^*$) by the conditions that:

1. $m(f_1 \otimes f_2, g_1 \otimes g_2) = m(f_1, g_1) \cdot m(f_2, g_2)$ for any $f_i, g_i \in \otimes^{k_i} V \otimes^{\ell_i} V^*$, $i = 1, 2$, with $k_1 + k_2 = k$, $\ell_1 + \ell_2 = \ell$,
2. $m(X_\flat, Y_\flat) = m(X, Y)$, where, for any $X \in V$, we define $X_\flat \in V^*$ by $X_\flat \doteq m(X, \cdot)$.

What are the components of this extension of m with respect to a basis of $\otimes^k V \otimes^\ell V^*$ associated to a basis $\{e_a\}_{\alpha=1}^{\dim V}$ of V?

(c) Let (V, m) be as in part (b). Prove that the extension of m to $\otimes^k V \otimes^\ell V^*$ is positive definite if m is positive definite. Is the analogous statement true if m is a Lorentzian inner product?

Solution. (a) The existence of such an inner product m on $V_1 \otimes V_2$ follows easily by fixing bases for V_1 and V_2: If $\{e_\alpha\}_{\alpha=1}^{\dim V_1}$ and $\{f_\beta\}_{\beta=1}^{\dim V_2}$ are bases (not necessarily orthonormal) for the vector spaces V_1 and V_2 respectively, and $\{e_*^\alpha\}_{\alpha=1}^{\dim V_1}$ and $\{f_*^\beta\}_{\beta=1}^{\dim V_2}$ are the corresponding *dual* bases for V_1^* and V_2^*, respectively (i.e. $e_*^\alpha(e_{\alpha'}) = \delta^\alpha_{\alpha'}$ and similarly for f_*^β, $f_{\beta'}$), one can easily check that the tensors $\{e_\alpha \otimes f_\beta\}_{\alpha,\beta}$ form a basis for $V_1 \otimes V_2$: Any $Z \in V_1 \otimes V_2$ (viewed as a bilinear map $Z : V_1^* \times V_2^* \to \mathbb{R}$ can be uniquely expressed as

$$Z = Z(e_*^\alpha, f_*^\beta) e_\alpha \otimes f_\beta \doteq Z^{\alpha\beta} e_\alpha \otimes f_\beta$$

by noting that, for any $v^* = v^*_\alpha e^\alpha_* \in V_1^*$ and $w^* = w^*_\beta f^\beta_* \in V_2^*$,

$$Z(v^*, w^*) = Z(v^*_\alpha e^\alpha_*, w^*_\beta f^\beta_*) = Z(e^\alpha_*, f^\beta_*) v^*_\alpha w^*_\beta = Z(e^\alpha_*, f^\beta_*) e_\alpha \otimes f_\beta(v^*, w^*).$$

With such bases fixed, let us define the inner product m on $V_1 \otimes V_2$ defined by

$$m(Z, W) = m\big(Z^{\alpha,\beta} e_\alpha \otimes f_\beta, W^{\alpha'\beta'} e_{\alpha'} \otimes f_{\beta'}\big) \doteq Z^{\alpha\beta} W^{\alpha'\beta'} m_1(e_\alpha, e_{\alpha'}) m_2(f_\beta, f_{\beta'})$$

(it is straightforward to check that m defined as above is symmetric and bilinear). Note that m satisfies the required property: For any $X_1, Y_1 \in V_1$ and $X_2, Y_2 \in V_2$:

$$\begin{aligned} m(X_1 \otimes X_2, Y_1 \otimes Y_2) &= m\big(X_1^\alpha X_2^\beta e_\alpha \otimes f_\beta,\, Y_1^{\alpha'} Y_2^{\beta'} e_{\alpha'} \otimes f_{\beta'}\big) \\ &= X_1^\alpha X_2^\beta Y_1^{\alpha'} Y_2^{\beta'} m_1(e_\alpha, e_{\alpha'}) m_2(f_\beta, f_{\beta'}) \\ &= m_1(X_1^\alpha e_\alpha, Y_1^{\alpha'} e_{\alpha'}) m_2(X_2^\beta f_\beta, Y_2^{\beta'} f_{\beta'}) \\ &= m_1(X_1, Y_1) m_2(X_2, Y_2). \end{aligned}$$

On the other hand, the uniqueness of m can be readily shown as follows: Let $\tilde{m}$ be another symmetric bilinear form satisfying

$$\tilde{m}(X_1 \otimes X_2, Y_1 \otimes Y_2) = m_1(X_1, Y_1) m_2(X_2, Y_2) = m(X_1 \otimes X_2, Y_1 \otimes Y_2) \quad \text{for all} \quad X_1, Y_1 \in V_1 \text{ and } X_2, Y_2 \in V_2$$

and let us consider the difference $M = m - \tilde{m}$. Then, the symmetric bilinear form M satisfies

$$M(e_\alpha \otimes f_\beta, e_{\alpha'} \otimes f_{\beta'}) = m_1(e_\alpha, e_{\alpha'}) m_2(f_\beta, f_{\beta'}) - m_1(e_\alpha, e_{\alpha'}) m_2(f_\beta, f_{\beta'}) = 0$$

for all $\alpha, \alpha', \beta, \beta'$; thus, since every element in $V_1 \otimes V_2$ can be written as a linear combination of tensors of the form $e_\alpha \otimes f_\beta$, we infer that $M \equiv 0$.

(b) We will first show that the two conditions indeed fix a unique symmetric bilinear form m on $\otimes^k V \otimes^l V^*$ for any $k, l \geqslant 0$. Arguing inductively on successive tensor products using part (a) of this exercise, it suffices to show that a unique such m is fixed for $k = 0, l = 1$, i.e. for V^*, by the condition that

$$m(X_\flat, Y_\flat) = m(X, Y) \quad \text{for all } X, Y \in V. \tag{1}$$

Note that, since m is assumed to be non-degenerate, for any $\omega \in V^*$ there exists a unique $\omega^\sharp \in V$ such that $\omega = m(\omega^\sharp, \cdot)$, i.e. $\omega = (\omega^\sharp)_\flat$. Thus, if we extend m on V^* by

$$m(\omega_1, \omega_2) \doteq m(\omega_1^\sharp, \omega_2^\sharp)$$

(note that the above expression is manifestly symmetric and bilinear in ω_1, ω_2), then (1) is satisfied.

As in part (a), let $\{e_\alpha\}_{\alpha=1}^{\dim V}$ be a basis of V and $\{e_*^\alpha\}_{\alpha=1}^{\dim V}$ be the corresponding dual basis of V^*. Let us denote with $m_{\alpha\beta} \doteq m(e_\alpha, e_\beta)$ the components of m with respect to the basis $\{e_\alpha\}_{\alpha=1}^{\dim V}$ of V and with $m^{\alpha\beta} \doteq m(e_*^\alpha, e_*^\beta)$ the corresponding components of m with respect to the basis $\{e_*^\alpha\}_{\alpha=1}^{\dim V}$ of $V*$. Then, the musical isomorphism $X \to X_\flat = m(X, \cdot)$ takes the form

$$(X_\flat)_\alpha = m_{\alpha\beta} X^\beta,$$

which implies in particular (in view of the fact that $(e_\alpha)^\beta = \delta_\alpha^\beta$)

$$\big((e_\gamma)_\flat\big)_\alpha = m_{\gamma\alpha}.$$

Our condition $m(X_\flat, Y_\flat) = m(X, Y)$ for the extension of m to V^* then yields for any e_α, e_β:

$$m((e_\alpha)_\flat, (e_\beta)_\flat) = m(e_\alpha, e_\beta) \Leftrightarrow m^{\gamma\delta}(e_\alpha)_\flat)_\gamma, (e_\beta)_\flat)_\delta = m_{\alpha\beta} \Leftrightarrow m^{\gamma\delta} m_{\gamma\alpha} m_{\delta\beta} = m_{\alpha\beta},$$

i.e. the matrix $[m^{\alpha\beta}]$ is the *inverse* of $[m_{\alpha\beta}]$. In particular, m is a non-degenerate inner product on V^* (since its matrix of coefficients is invertible).

Arguing inductively using part (a), we infer that m admits a unique extension with the required property on $\otimes^k V \otimes^l V^*$ for any $k, l \geqslant 0$. An expression for m with respect to the coordinate basis $\left\{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes e_*^{\beta_1} \otimes \cdots \otimes e_*^{\beta_l}\right\}_{\alpha_1, \ldots, \beta_1, \ldots = 1}^{\dim V}$ of $\otimes^k V \otimes^l V^*$ can be readily obtained using our condition on the factorization of $m(\cdot, \cdot)$ when acting on tensors of rank 1: If we denote with

$$m_{\alpha_1 \ldots a_l a_1' \ldots a_l'}^{\beta_1 \ldots \beta_k \beta_1' \ldots \beta_k'} = m\big(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes e_*^{\beta_1} \otimes \cdots \otimes e_*^{\beta_l},\, e_{\alpha_1'} \otimes \cdots \otimes e_{\alpha_k'} \otimes e_*^{\beta_1'} \otimes \cdots \otimes e_*^{\beta_l'}\big)$$

the coeffcients of the extension m with respect to the above basis, we calculate:

$$\begin{aligned} m^{\beta_1\ldots\beta_k\beta'_1\ldots\beta'_k}_{\alpha_1\ldots a_l a'_1\ldots a'_l} &= m\big(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_k}\otimes e_*^{\beta_1}\otimes\cdots\otimes e_*^{\beta_l},\, e_{\alpha'_1}\otimes\cdots\otimes e_{\alpha'_k}\otimes e_*^{\beta'_1}\otimes\cdots\otimes e_*^{\beta'_l}\big)\\ &= m(e_{\alpha_1},e_{\alpha'_1})\ldots m(e_{\alpha_k},e_{\alpha'_k})m(e_*^{\beta_1},e_*^{\beta'_1})\ldots m(e_*^{\beta_l},e_*^{\beta'_l})\\ &= m_{\alpha_1\alpha'_1}\ldots m_{\alpha_k\alpha'_k}m^{\beta_1\beta'_1}\ldots m^{\beta_l\beta'_l}. \end{aligned}$$

The fact that m on $\otimes^k V\otimes^l V^*$ is non-degenerate follows readily from the fact that the corresponding inner products on V and V^* are non-degenerate: Let $X\in\otimes^k V\otimes^l V^*$ be such that

$$m(X,Y)=0 \quad \text{for all} \quad Y\in\otimes^k V\otimes^l V^*.$$

In particular, expanding $X=X^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_k}\otimes e_*^{\beta_1}\otimes\cdots\otimes e_*^{\beta_l}$, we obtain for $Y=e_{\alpha'_1}\otimes\cdots\otimes e_{\alpha'_k}\otimes e_*^{\beta'_1}\otimes\cdots\otimes e_*^{\beta'_l}$

$$\begin{aligned} &m\big(X,e_{\alpha'_1}\otimes\cdots\otimes e_{\alpha'_k}\otimes e_*^{\beta'_1}\otimes\cdots\otimes e_*^{\beta'_l}\big)=0\\ &\Rightarrow X^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}m_{\alpha_1\alpha'_1}\ldots m_{\alpha_k\alpha'_k}m^{\beta_1\beta'_1}\ldots m^{\beta_l\beta'_l}=0 \quad \text{for all } \alpha'_1,\ldots\alpha'_k,\beta'_1,\ldots\beta'_l\in\{1,\ldots,\text{dim}V\}. \end{aligned}$$

Since the matrices $[m_{\alpha\alpha'}]$ and $[m^{\beta\beta'}]$ are invertible, we infer from the above (for instance by multiplying with $m^{\alpha'_1\gamma_1}\ldots m^{\alpha'_k\gamma_k}m_{\beta'_1\delta_1}\ldots m_{\beta'_l,\delta_l}$) that

$$X^{\gamma_1\ldots\gamma_k}_{\delta_1\ldots\delta_l}=0 \quad \text{for all } \gamma_1,\ldots,\gamma_k,\delta_1,\ldots,\delta_l\in\{1,\text{dim}V\},$$

i.e. that $X=0$.

(c) In the case when m is a positive definite inner product on V, let us assume that $\{e_\alpha\}_{\alpha=1}^{\text{dim}V}$ is an *orthonormal* basis of V, so that $m_{\alpha\beta}=\delta_{\alpha\beta}$. In that case, the dual basis $\{e_*^\alpha\}_{\alpha=1}^{\text{dim}V^*}$ of V^* is also orthonormal (since $m^{\alpha\beta}=[m_{\alpha\beta}]^{-1}=\delta_{\alpha\beta}$). Hence, for any $X\in\otimes^k V\otimes^l V^*$, we compute

$$m(X,X)=m_{\alpha_1\alpha'_1}\ldots m_{\alpha_k\alpha'_k}m^{\beta_1\beta'_1}\ldots m^{\beta_l\beta'_l}X^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}X^{\alpha'_1\ldots\alpha'_k}_{\beta'_1\ldots\beta'_l}=\sum_{\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l=1}^{\text{dim}V}(X^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l})^2,$$

so the extended inner product on $\otimes^k V\otimes^l V^*$ is also positive definite.

In the case when (V,m) is a Lorentzian inner product space, let T be a timelike vector of V and X a (non-zero) spacelike vector with $T\perp X$. Then, it is easy to verify that $T\otimes X$ and $X\otimes T$ are linearly independent $(2,0)$ tensors and, moreover, any tensor of the form

$$V=\lambda_1 T\otimes X+\lambda_2 X\otimes T, \quad (\lambda_1,\lambda_2)\in\mathbb{R}^2\setminus 0$$

satisfies

$$\begin{aligned} m(V,V)&=\lambda_1^2 m(T\otimes X,T\otimes X)+2\lambda_1\lambda_2 m(T\otimes X,X\otimes T)+\lambda_2^2 m(X\otimes T,X\otimes T)\\ &=\lambda_1^2 m(T,T)m(X,X)+2\lambda_1\lambda_2 m(T,X)m(X,T)+\lambda_2^2 m(X,X)m(T,T)\\ &=(\lambda_1^2+\lambda_2^2)m(T,T)m(X,X)<0. \end{aligned}$$

Hence, $(\otimes^2 V,m)$ is not a Lorentzian inner product space, since m restricts to a negative definite inner product on a 2-dimensional subspace of $\otimes^2 V$.

3.2 Let $\mathcal{M}^n$ be a smooth manifold and let $\omega : \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$ be $C^\infty(\mathcal{M})$-linear functional. We will show that ω is in fact an 1-form on $\mathcal{M}$, i.e. if $Y \in \Gamma(\mathcal{M})$ then, for all $p \in \mathcal{M}$, $\omega(Y)|_p$ depends only on $Y|_p$.

(a) Let $\mathcal{U}$ be an open neighborhood of p covered by a coordinate chart $(x^1, \dots, x^n)$. Show that there exists an open neighborhood $\mathcal{V}$ of p contained inside $\mathcal{U}$ and smooth vector fields $\{X_i\}_{i=1}^n$ on $\mathcal{M}$ such that $X_i = \frac{\partial}{\partial x^i}$ on $\mathcal{V}$.

(b) Show that if $Y|_p = 0$, then there exists a finite number of vector fields $\{V_k\}_k$ such that

$$Y = \sum_k f_k V_k,$$

where the functions $f_k \in C^\infty(\mathcal{M})$ satisfy $f_k(p) = 0$. Deduce that $\omega(Y)|_p = 0$ and, more generally, $\omega(Y)|_p$ depends only on $Y|_p$.

The same argument should also work for more general $C^\infty(\mathcal{M})$-multilinear maps $T : \Gamma^*(\mathcal{M}) \times \cdots \times \Gamma^*(\mathcal{M}) \times \Gamma(\mathcal{M}) \times \cdots \times \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$.

Solution. (a) Let $\phi : \mathcal{U} \to \mathbb{R}^n$ be a local coordinate chart defined on a neighborhood $\mathcal{U}$ of p and let $(x^1, \dots, x^n)$ be the associated coordinate functions. Since $\phi(\mathcal{U})$ is an open subset of $\mathbb{R}^n$, there exists a radius $r > 0$ so that the Euclidean ball $B_{3r}(\phi(p))$ of radius $3r$ centered at $\phi(p)$ is entirely contained in $\phi(\mathcal{U})$. Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function so that

$$\chi \equiv 1 \text{ on } B_r(\phi(p)) \text{ and } \chi \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{2r}(\phi(p)).$$

Let us set $\mathcal{V}_r = \phi^{-1}\big(B_r(\phi(p))\big)$, $\mathcal{V}_{2r} = \phi^{-1}\big(B_{2r}(\phi(p))\big)$ and $\mathcal{V}_{3r} = \phi^{-1}\big(B_{3r}(\phi(p))\big)$ (see Figure 1). Notice that, since ϕ is a homeomorphism, these are open subsets of $\mathcal{M}$, satisfying

$$p \in \mathcal{V}_r \subset \mathcal{V}_{2r} \subset \mathcal{V}_{3r}.$$

Moreover, since $\mathrm{clos}\big(B_{2r}(\phi(p))\big)$ is a compact subset of $\phi(\mathcal{U})$ (since it is strictly contained inside $B_{3r}(\phi(p)) \subset \phi(\mathcal{U})$) and $\phi^{-1} : \phi(\mathcal{U}) \to \mathcal{U}$ is a homeomorphism, we know that $\mathrm{clos}\big(B_{2r}(\phi(p))\big)$ is a compact (and, hence, closed) subset of $\mathcal{U}$. Since $\mathcal{U}$ is open, this implies in particular that

$$\partial\mathcal{U} \cap \mathrm{clos}\big(B_{2r}(\phi(p))\big) = \emptyset. \tag{2}$$

Let us define the function $\psi : \mathcal{M} \to \mathbb{R}$ by the relation

$$\psi(q) = \begin{cases} \chi \circ \phi(q), & \text{if } q \in \mathcal{U}, \\ 0, & \text{if } q \in \mathcal{M} \setminus \mathcal{U}. \end{cases}$$

Note that the support of ψ is contained in the set $\mathcal{V}_{2r}$ and $\psi \equiv 1$ on $\mathcal{V}_r$. We will now show that ψ is a smooth function on $\mathcal{M}$. The definition of ψ implies that it is automatically smooth in the open sets $\mathcal{U}$ and $\mathrm{int}\big(\mathcal{M} \setminus \mathcal{U}\big)$; thus, we only have to check its behaviour at $\partial\mathcal{U}$. It will follow that

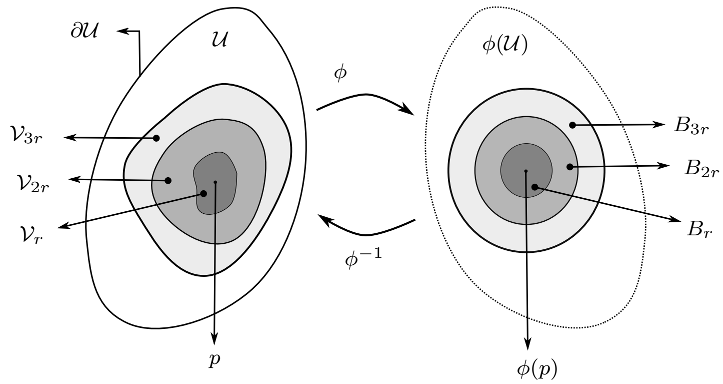

Figure 1: Schematic depiction of the subsets $\mathcal{V}_r$, $\mathcal{V}_{2r}$, $\mathcal{V}_{3r} \subset \mathcal{U}$ and $B_r(\phi(p))$, $B_{2r}(\phi(p))$, $B_{3r}(\phi(p)) \subset \mathbb{R}^n$. Note that the function ψ is supported in $\mathcal{V}_{2r}$ and $\psi \equiv 1$ on $\mathcal{V}_r$.

$\psi \in C^\infty(\mathcal{M})$ if the set $\mathcal{Z} = \{q \in \mathcal{M} : \psi(q) = 0\}$ contains an open neighborhood of $\partial\mathcal{U}$. Indeed, since ψ is supported in V_{2r}, the set $\mathcal{Z}$ contains the open set $\mathcal{W} = \mathcal{M} \setminus \mathrm{clos}(V_{2r})$ and, in view of (2),

$$\partial\mathcal{U} \subset \mathcal{W}.$$

Having defined the smooth cut-off function $\psi : \mathcal{M} \to \mathbb{R}$ as above, let us define the vector fields X_i $(i = 1, \dots, n)$ on $\mathcal{M}$ as follows:

$$(X_i)|_q = \begin{cases} \psi(q)\frac{\partial}{\partial x^i}, & \text{if } q \in \mathcal{U}, \\ 0, & \text{if } q \in \mathcal{M} \setminus \mathcal{U}. \end{cases}$$

The vector fields X_i are indeed smooth for the same reason that ψ is smooth: They are trivially smooth on $\mathcal{U}$ and $\mathrm{int}(\mathcal{M} \setminus \mathcal{U})$ and, since ψ vanishes on an open neighborhood of $\partial\mathcal{U}$, they are equal to the zero vector field in a neighborhood of $\partial\mathcal{U}$ (and hence they are also smooth at $\partial\mathcal{U}$). Moreover, since $\psi = 1$ on $\mathcal{V}_r$, we have

$$X_i = \frac{\partial}{\partial x^i} \quad \text{on the neighborhood } \mathcal{V}_r \text{ of } p.$$

(b) Let $Y \in \Gamma(\mathcal{M})$ be such that $Y|_p = 0$. Note that, inside the open neighborhood $\mathcal{U}$ of p covered by the coordinates $(x^1, \dots, x^n)$, we can easily write Y as a sum of vector fields with coefficients vanishing at p, since

$$Y = Y^i \frac{\partial}{\partial x^i}$$

and $Y^1(p) = \cdots = Y^n(p) = 0$ since $Y|_p = 0$. The challenge is to obtain a similar decomposition which is valid on the whole of $\mathcal{M}$ (where $\frac{\partial}{\partial x^i}$ is not well defined). To this end, we will use the cut-off function ψ and the vector fields X_i from part (b) of the exercise.

Let us first decompose (trivially)

$$Y = \psi^2 Y + (1 - \psi^2)Y. \tag{3}$$

If Y^i are the components of the vector field Y in the coordinate system $(x^1, \dots, x^n)$ on $\mathcal{U}$, then the vector field ψY can be expressed as

$$\psi(q)Y|_q = \psi(q)Y^i(q)\frac{\partial}{\partial x^i} = Y^i(q)X_i|_q \text{ for all } q \in \mathcal{U}.$$

Therefore, we have

$$\psi^2(q)Y|_q = \big(\psi Y^i\big)(q) \cdot X_i|_q \text{ for all } q \in \mathcal{U}. \tag{4}$$

Notice that, in the above expression, the vector fields $\psi^2 Y$ and X_i are defined on the whole of the manifold $\mathcal{M}$, but the functions ψY^i are only defined on $\mathcal{U}$ (covered by the coordinate system $(x^1, \dots, x^n)$). However, for each $i = 1, ldots, n$, ψY^i vanishes in an open neighborhood of $\partial\mathcal{U}$ and hence (as in the case of ψ) it can be extended as a smooth function $h^i \in C^\infty(\mathcal{M})$ so that

$$h^i(q) = \Big\{\psi(q)Y^i(q), \quad \text{if } q \in \mathcal{U}, 0, \quad \text{if } q \in \mathcal{M} \setminus \mathcal{U}.$$

Then, since the vector field $\psi^2 Y$ satisfies (4) on $\mathcal{U}$ and vanishes identically on $\mathcal{M} \setminus \mathcal{U}$, we have

$$\psi^2 Y = h^i X_i \quad \text{everywhere on } \mathcal{M}.$$

Returning to (3), we have

$$Y = h^i X_i + (1 - \psi^2)Y.$$

Notice that, on the right hand side, the coefficient of each vector field vanishes at p:

- For $i = 1, \dots, n$, $h^i(p) = Y^i(p) = 0$ since we assumed that $Y|_p = 0$.
- $(1 - \psi^2)(p) = 0$ since $\psi(p) = 1$.

Thus, we succeeded to write

$$Y = \sum_k f_k V_k$$

for $f_k \in C^\infty(\mathcal{M})$ and $V_k \in \Gamma(\mathcal{M})$ such that $f_k(p) = 0$.

In view of our assumption that $\omega(\cdot)$ is $C^\infty(\mathcal{M})$ in its argument, we therefore have:

$$\big(\omega(Y)\big)(p) = \Big(\omega\big(\sum_k f_k V_k\big)\Big)(p) = \sum_k f_k(p)\big(\omega(V_k)\big)(p) = 0.$$

By linearity, we also deduce that if $Y_1, Y_2 \in \Gamma(\mathcal{M})$ satisfy $Y_1|_p = Y_2|_p$, then

$$\big(\omega(Y_1)\big)(p) - \big(\omega(Y_2)\big)(p) = \big(\omega(Y_1 - Y_2)\big)(p) = 0.$$

3.3 Let $\mathcal{M}^n$ be a smooth manifold and let $(x^1, \dots, x^n)$ a local system of coordinates around $p \in \mathcal{M}$. Let also $S \in \otimes^k T_p\mathcal{M} \otimes^l T_p^*\mathcal{M}$ be a tensor of type (k, l) at p and let $S^{i_1 i_2 \dots i_k}{}_{i_1 j_2 \dots j_l}$ be its corresponding components. We will define the *contraction* $\mathrm{tr}(S)$ to be the tensor

$$\mathrm{tr}(S) = S^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_l},$$

i.e. the components of $\mathrm{tr}(S)$ in the $(x^1, \dots, x^n)$ coordinates are simply the components of S after summing over the first covariant and contravariant indices. Show that $\mathrm{tr}(S)$ is well-defined *independently* of the choice of coordinate system, i.e. show that if $(y^1, \dots, y^n)$ is a different coordinate system around p and $\tilde{S}^{i_1 i_2 \dots i_k}{}_{j_1 j_2 \dots j_l}$ are the components of S with respect to these coordinates, then

$$S^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_l} = \tilde{S}^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial y^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_k}} \otimes dy^{i_2} \otimes \dots \otimes dy^{i_l}.$$

Solution. Let $S^{i_1 \dots i_k}{}_{j_1 \dots j_l}$ and $\tilde{S}^{i_1 \dots i_k}{}_{j_1 \dots j_l}$ be the components of S in the $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ coordinate systems, respectively. The two sets of coordinate tangent vectors and cotangent vectors are related by

$$\frac{\partial}{\partial y^i} = \frac{\partial x^a}{\partial y^i} \quad \text{and} \quad dy^i = \frac{\partial y^i}{\partial x^a} dx^a,$$

while the relation between the two sets of components for S is given by the usual transformation law for tensors, i.e.

$$\tilde{S}^{i_1 \dots i_k}{}_{j_1 \dots j_l} = S^{a_1 \dots a_k}{}_{b_1 \dots b_l} \frac{\partial y^{i_1}}{\partial x^{a_1}} \dots \frac{\partial y^{i_k}}{\partial x^{a_k}} \frac{\partial x^{b_1}}{\partial y^{j_1}} \dots \frac{\partial x^{b_l}}{\partial y^{j_l}}. \tag{5}$$

In the above, $\frac{\partial y^i}{\partial x^a}$ denotes the Jacobian matrix of $y = (y^1, \dots, y^n)$ as a function of $x = (x^1, \dots, x^1)$ (see the 1^{st} Exercise Series), while $\frac{\partial x^a}{\partial y^i}$ denotes the Jacobian of the inverse function $x = x(y)$. Recall that, for any diffeomorphism $\Phi : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n$, the Jacobian matrix $\big[D(\Phi^{-1})\big]$ of the inverse function Φ^{-1} satisfies:

$$\big[D(\Phi^{-1})\big]\big(\Phi(z)\big) = \big[D(\Phi^{-1})\big]^{-1}(z) \quad \text{for all } z \in \Omega.$$

Therefore, as we've seen in class, the matrices $\left[\frac{\partial y^i}{\partial x^a}\right]$ and $\left[\frac{\partial x^a}{\partial y^i}\right]$ evaluated at the same point p in the common domain of definition of the coordinate charts $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ are the inverse of one another, i.e.

$$\frac{\partial y^i}{\partial x^a} \cdot \frac{\partial x^a}{\partial y^j} = \delta^i_j \quad \text{and} \quad \frac{\partial x^a}{\partial y^i} \cdot \frac{\partial y^i}{\partial x^b} = \delta^a_b. \tag{6}$$

In order for the contraction $\mathrm{tr}(S)$ to be well-defined independently of the coordinate system, we need to show that

$$S^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_l} = \tilde{S}^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial y^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_k}} \otimes dy^{i_2} \otimes \dots \otimes dy^{i_l},$$

which is the same as saying that the components of $\mathrm{tr}(S)$ transform under changes of coordinates like a tensor of type $(k-1, l-1)$, i.e.:

$$\mathrm{tr}(\tilde{S})^{i_2 \dots i_k}{}_{j_2 \dots j_l} = \mathrm{tr}(\tilde{S})^{a_2 \dots a_k}{}_{b_2 \dots b_l} \frac{\partial y^{i_2}}{\partial x^{a_2}} \dots \frac{\partial y^{i_k}}{\partial x^{a_k}} \frac{\partial x^{b_2}}{\partial y^{j_2}} \dots \frac{\partial x^{b_l}}{\partial y^{j_l}}. \tag{7}$$

In order to show (7), we will calculate $\mathrm{tr}(\tilde{S})$ using the formula (5):

$$\begin{aligned}
\mathrm{tr}(\tilde{S})^{i_2\ldots i_k}{}_{j_2\ldots j_l} &= \tilde{S}^{\alpha i_2\ldots i_k}{}_{\alpha j_2\ldots j_l} \\
&= S^{a_1\ldots a_k}{}_{b_1 b_2\ldots b_l}\frac{\partial y^\alpha}{\partial x^{a_1}}\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_1}}{\partial y^\alpha}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}} \\
&= S^{a_1 a_2\ldots a_k}{}_{b_1 b_2\ldots b_l}\left(\frac{\partial y^\alpha}{\partial x^{a_1}}\cdot\frac{\partial x^{b_1}}{\partial y^\alpha}\right)\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}} \\
&\overset{(6)}{=} S^{a_1 a_2\ldots a_k}{}_{b_1 b_2\ldots b_l}\cdot\delta^{b_1}_{a_1}\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}} \\
&= S^{\alpha a_2\ldots a_k}{}_{\alpha b_2\ldots b_l}\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}} \\
&= \mathrm{tr}(S)^{a_2\ldots a_k}{}_{b_2\ldots b_l}\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}},
\end{aligned}$$

i.e. (7) holds.

3.4 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold which is *not* time orientable. Prove that there exists a Lorentzian manifold $(\mathcal{M}', g')$ which is time orientable and a map $F : M' \to M$ which is $2-1$ and a local isometry. Such a space is called a *time-orientable cover*. (*Hint: You might want to consider the causal line seed field* $\{X, -X\}$ *over* $\mathcal{M}$ *constructed in Exercise 2.4 last week, and study its properties a a submanifold of* $T\mathcal{M}$.)

Solution. We have seen in class that a Lorentzian manifold $(\mathcal{M}, g)$ is time orientable if and only if there exists a causal vector field $X \in \Gamma(\mathcal{M})$. We also saw in Exercise 2.4 that *any* Lorentzian manifold $(\mathcal{M}, g)$ (whether time-orientable or not) admits a smooth *causal line field*, that is to say, an assignment of a pair of opposite tangent vectors $p \to \mathscr{S}_p = \{X_p, -X_p\} \subset T_p\mathcal{M} \setminus 0$ for all $p \in \mathcal{M}$ such that, for each $p \in \mathcal{M}$:

1. The vectors $X_p, -X_p \in T_p\mathcal{M} \setminus 0$ are causal with respect to g_p,
2. There exists an open neighborhood $\mathcal{U}_p$ and a smooth vector field Y on $\mathcal{U}$ such that, for all $q \in \mathcal{U}$, $\mathscr{S}_q = \{Y_q, -Y_q\}$ (note that such a vector field Y cannot exist globally on $\mathcal{M}$ if $(\mathcal{M}, g)$ is not time orientable).

Let us consider the subset $\mathscr{S}$ of $T\mathcal{M}$ defined by

$$\mathscr{S} = \bigcup_{p\in\mathcal{M}} \mathscr{S}_p \subset \bigcup_{p\in\mathcal{M}} T_p\mathcal{M} = T\mathcal{M}.$$

We will first show that $\mathscr{S}$ is a smooth submanifold of $T\mathcal{M}$. To this end, it suffices to show that, for any $p \in \mathcal{M}$, there exists an open neighborhood $\mathcal{V}$ of p such that $\pi^{-1}(\mathcal{V}) \cap \mathscr{S}$ is a submanifold of $T\mathcal{M}$; recall that $\pi : T\mathcal{M} \to \mathcal{M}$ is the base projection map

$$\pi(q, \xi) = q \quad \text{for any } q \in \mathcal{M}, \xi \in T_q\mathcal{M}.$$

For any point $p \in \mathcal{M}$, property 2 above says that there exists an open neighorhood $\mathcal{U}$ of p and a vector field Y on $\mathcal{U}$ such that, if we view Y as a map from $\mathcal{U}$ to $T\mathcal{U}$ (sending $p \to Y_p \in T_p\mathcal{M}$), then $\mathscr{S}|_{\mathcal{U}} = \mathscr{S} \cap \pi^{-1}(\mathcal{U})$ is just the disjoint union of the images of Y and $-Y$, i.e.

$$\mathscr{S}|_{\mathcal{U}} = Y(\mathcal{U}) \coprod \Big(- Y(\mathcal{U})\Big)) \doteq \mathscr{S}_+(\mathcal{U}) \coprod \mathscr{S}_-(\mathcal{U}).$$

Given any local coordinate chart $\Phi : \mathcal{V} \to \mathbb{R}^n$ on an open set $\mathcal{V} \subset \mathcal{U}$ with associated coordinates $(x^1, \dots, x^n)$, we can define a coordinate chart $\tilde{\Phi} : T\mathcal{V} \to \mathbb{R}^{2n}$ with associated coordinates $(x^1, \dots, x^n;\, v^1, \dots, v^n)$ so that, for any $p \in \mathcal{V}$ and $\xi \in T_p\mathcal{M}$:

$$\big(x^1, \dots, x^n;\, v^1, \dots, v^n\big)(p, \xi) = \big(x^1(p), \dots, x^n(p);\, dx^1|_p(\xi), \dots, dx^n|_p(\xi)\big).$$

In any such coordinate system $(x^1, \dots, x^n;\, v^1, \dots, v^n)$, the sets $\mathscr{S}_\pm(\mathcal{V})$ correspond to the smooth submanifolds of $\mathbb{R}^{2n}$ described by the equations

$$v^i = \pm Y^i(x^1, \dots, x^n), \quad i = 1, \dots, n.$$

Thus, $\mathscr{S}$ is a smooth submanifold of $T\mathcal{M}$. Moreover, for $\mathcal{V}$ as above, the maps $Y : \mathcal{V} \to \mathscr{S}_+(\mathcal{V})$ and $-Y : \mathcal{V} \to \mathscr{S}_-(\mathcal{V})$ are diffeomorphisms: They are immersions (since any vector field $Y : \mathcal{V} \to T\mathcal{V}$ is an immersion, as can be explicitly checked in the coordinates fixed above) and they satisfy

$$\pi \circ Y = \mathrm{Id}_{\mathcal{V}}, \quad \pi \circ (-Y) = \mathrm{Id}_{\mathcal{V}}.$$

As a result, the map $\pi : \mathscr{S} \to \mathcal{M}$ is a local diffeomorphism (not a global one, though, since the inverse image of any point of $\mathcal{M}$ contains two points of $\mathscr{S}$). We can therefore equip $\mathscr{S}$ with the pull-back metric $g' = \pi_* g$ (this is a well-defined Lorentzian metric, since $d\pi : T_w\mathscr{S} \to T_{\pi(w)}\mathcal{M}$ is 1-1 and onto for any $w \in \mathscr{S}$); this, by definition, turns the map $\pi : (\mathscr{S}, g') \to (\mathcal{M}, g)$ into a local isometry.

We will now show that $(\mathscr{S}, g')$ is time-orientable. To this end, it suffices to find a globally defined smooth causal vector field on $\mathscr{S}$. From our definition of $\mathscr{S}$, any point $w \in \mathscr{S} \subset T\mathcal{M}$ is of the form (q, ξ) for some $q \in \mathcal{M}$ and $\xi \in T_q\mathcal{M} \setminus 0$ which is *causal* with respect to g_q. Since $\pi : \mathscr{S} \to \mathcal{M}$, $\pi(q, \xi) = q$, is a local isometry, the differential $d\pi|_{(q,\xi)} : (T_{(q,\xi)}\mathscr{S}, g'|_{(q,\xi)}) \to (T_q\mathcal{M}, g|_q)$ is a linear isometry; thus, we can define the vector field Y' on $\mathscr{S}$ by the relation

$$Y'|_{(q,\xi)} = \big(d\pi|_{(q,\xi)}\big)^{-1}\xi \quad \text{for any } (q, \xi) \in \mathscr{S}. \tag{8}$$

Note that, since $d\pi|_{(q,\xi)}\big)^{-1}$ is a linear isometry, $Y'|_{(q,\xi)}$ is causal with respect to g' (since ξ is causal with respect to g). Moreover, Y' as defined above is indeed smooth since, for any $w = (q, \xi) \in \mathscr{S}$, there exists an open neighborhood $\mathcal{U}'$ of w in $\mathscr{S}$ such that $\pi : \mathcal{U}' \to \pi(\mathcal{U}')$ is a diffeomorphism and any $w' = (q', \xi') \in \mathcal{U}'$ is of the form $\xi' = Y|_{q'}$ for a smooth vector field Y on $\pi(\mathcal{U}')$ (this is essentially property 2 above); thus, $Y'|_{\mathcal{U}'}$ as defined by (8) is the push-forward of the smooth vector field Y on $\pi(\mathcal{U}') \subset \mathcal{M}$ via the map $Y : \pi(\mathcal{U}') \to \mathcal{U}'$ (viewed as the inverse of $\pi|_{\mathcal{U}'}$) and, therefore, $Y'|_{\mathcal{U}'}$ is smooth.

We have, thus, shown that $\pi : (\mathscr{S}, g') \to (\mathcal{M}, g)$ is a $2 - 1$ map which is a local isometry and that $(\mathscr{S}, g')$ is time-orientable; this construction is carried out irrespectively of whether $(\mathcal{M}, g)$ is time orientable or not. In the case when $(\mathcal{M}, g)$ is time orientable, $\mathscr{S}$ will consist of two components (since the causal line field in this case can be written as the union of two causal vector fields defined everywhere on $\mathcal{M}$) and π is an isometry when restricted to each of them. If $(\mathcal{M}, g)$, then $\mathscr{S}$ is connected.